

Hypersurface of a Finsler space subjected to an h -exponential change of metric

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Abstract

Recently we have obtained the Cartan connection for the Finsler space whose metric is given by an exponential change with an h -vector. In this paper, we discuss certain geometric properties of a Finslerian hyperspace subjected to an h -exponential change of metric.

Keywords : Finsler space, hypersurface, exponential change, h -vector.

2000 Mathematics Subject Classification : **53B40**.

1 Introduction

In 2006, YU Yao-yong and YOU Ying [12] studied a Finsler space with metric function given by exponential change of Riemannian metric. In 2012, H. S. Shukla et. al. [10] considered a Finsler space $\overline{F}^n = (M^n, \overline{L})$, whose Fundamental metric function is an exponential change of Finsler metric function given by

$$\overline{L} = L e^{\beta/L},$$

where $\beta = b_i(x)y^i$ is 1-form on manifold M^n .

H. Izumi [7] introduced the concept of an h -vector $b_i(x, y)$ which is v-covariant constant with respect to the Cartan connection and satisfies $L C_{ij}^h b_h = \rho h_{ij}$, where ρ is a non-zero scalar function and C_{jk}^i are components of Cartan tensor. Thus if b_i is an

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h -vector, then

$$(1.1) \quad (i) \ b_i|_k = 0, \quad (ii) \ L C_{ij}^h b_h = \rho h_{ij}.$$

From the above definition, we have

$$(1.2) \quad L \dot{\partial}_j b_i = \rho h_{ij},$$

which shows that b_i is a function of directional argument also. H. Izumi [7] proved that the scalar ρ is independent of directional argument. Gupta and Pandey [6] proved that if the h -vector b_i is gradient then the scalar ρ is constant

B. N. Prasad [9] obtained the Cartan connection of Finsler space whose metric is given by h -Rander's change of a Finsler metric. Gupta and Pandey [6] obtained the Cartan connection of Finsler space whose metric is given by h -Kropina change of Finsler metric. Present authors [2] studied the Cartan connection of Finsler space whose metric is given by h -exponential change of Finsler metric.

The theory of hypersurfaces in a Finsler space has been introduced by E. Cartan [1]. A. Rapcsák [11] introduced three kinds of hyperplanes and M. Matsumoto [8] has classified the hypersurfaces and developed a systematic theory of Finslerian hypersurfaces. Gupta and Pandey [4, 5] discussed the hypersurface of a Finsler space whose metric is given by certain transformation with an h -vector.

In the present paper, we discuss the geometric properties of hypersurface of a Finsler space ${}^*F^n = (M^n, {}^*L)$, whose metric function *L is given by an h -exponential change of a Finsler metric function *i.e.*

$$(1.3) \quad {}^*L = L e^{\frac{\beta}{L}},$$

where $\beta = b_i(x, y)y^i$ and b_i is an h -vector.

2 Preliminaries

Let $F^n = (M^n, L)$ be an n -dimensional Finsler space equipped with the Fundamental function $L(x, y)$. The metric tensor, angular metric tensor and Cartan tensor are defined by $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2$, $h_{ij} = g_{ij} - l_i l_j$ and $C_{ijk} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^2$ respectively, where $\dot{\partial}_k = \frac{\partial}{\partial y^k}$. The

Cartan connection is given by $CT = (F_{jk}^i, N_k^i, C_{jk}^i)$. The h - and v -covariant derivatives $X_{i|j}$ and $X_i|_j$ of a covariant vector field X_i are defined by

$$(2.1) \quad X_{i|j} = \partial_j X_i - N_j^r \dot{\partial}_r X_i - X_r F_{ij}^r,$$

and

$$(2.2) \quad X_i|_j = \dot{\partial}_j X_i - X_r C_{ij}^r,$$

where $\partial_k = \frac{\partial}{\partial x^k}$.

A hypersurface M^{n-1} of the underlying smooth manifold M^n may be parametrically represented by the equation $x^i = x^i(u^\alpha)$, where u^α are Gaussian coordinates on M^{n-1} (Latin indices run from 1 to n while Greek indices run from 1 to $n-1$). Here, we shall assume that the matrix consisting of the pojection factors $B_\alpha^i = \partial x^i / \partial u^\alpha$ is of rank $n-1$. If the supporting element y^i at a point $u = (u^\alpha)$ of M^{n-1} is assumed to be tangent to M^{n-1} , we may then write $y^i = B_\alpha^i(u) v^\alpha$ so that $v = (v^\alpha)$ is thought of as the supporting element of M^{n-1} at a point u^α . Since the function $\underline{L}(u, v) = L(x(u), y(u, v))$ gives arise a Finsler function on M^{n-1} , we get an $(n-1)$ -dimensional Finsler space $F^{n-1} = (M^{n-1}, \underline{L}(u, v))$.

At each point u^α of F^{n-1} , the unit normal vector $N^i(u, v)$ is defined as

$$(2.3) \quad g_{ij} B_\alpha^i N^j = 0 \quad g_{ij} N^i N^j = 1,$$

The inverse projection factors $B_i^\alpha(u, v)$ of B_α^i are defined as

$$(2.4) \quad B_i^\alpha = g^{\alpha\beta} g_{ij} B_\beta^j,$$

where $g^{\alpha\beta}$ is the inverse of metric tensor $g_{\alpha\beta}$ of F^{n-1} .

from (2.3) and (2.4), it follows that

$$(2.5) \quad B_\alpha^i B_i^\beta = \delta_\alpha^\beta, \quad B_\alpha^i N_i = 0, \quad N_i B_\alpha^i = 0, \quad N_i N^i = 1,$$

and further

$$(2.6) \quad B_\alpha^i B_j^\beta + N^i N_j = \delta_j^\beta.$$

For the induced Cartan connection $ICT = (F_{\beta\gamma}^\alpha, G_\beta^\alpha, C_{\beta\gamma}^\alpha)$ on F^{n-1} , the second fundamental h -tensor $H_{\alpha\beta}$ and the normal curvature vector H_α are given by

$$(2.7) \quad H_{\alpha\beta} = N_i (B_{\alpha\beta}^i + F_{jk}^i B_\alpha^j B_\beta^k) + M_\alpha H_\beta,$$

and

$$(2.8) \quad H_\alpha = N_i(B_{0\alpha}^i + G_j^i B_\alpha^j),$$

where $M_\alpha = C_{ijk} B_\alpha^i N^j N^k$, $B_{\alpha\beta}^i = \frac{\partial^2 x^i}{\partial u^\alpha \partial v^\beta}$ and $B_{0\alpha}^i = B_{\beta\alpha}^i v^\beta$.

The equations (2.7) and (2.8) yield

$$(2.9) \quad H_{0\alpha} = H_{\beta\alpha} v^\beta = H_\alpha, \quad H_{\alpha 0} = H_{\alpha\beta} v^\beta = H_\alpha + M_\alpha H_0.$$

The second fundamental v-tensor $M_{\alpha\beta}$ is defined as

$$(2.10) \quad M_{\alpha\beta} = C_{ijk} B_\alpha^i B_\beta^j N^k.$$

The relative h - and v -covariant derivatives of B_α^i and N^i are given by

$$(2.11) \quad \begin{aligned} B_{\alpha|\beta}^i &= H_{\alpha\beta} N^i, & B_\alpha^i|_\beta &= M_{\alpha\beta} N^i, \\ N_{|\beta}^i &= -H_{\alpha\beta} B_j^\alpha g^{ij}, & N^i|_\beta &= -M_{\alpha\beta} B_j^\alpha g^{ij}. \end{aligned}$$

Let $X_i(x, y)$ be a vector field on F^n . Then the relative h - and v -covariant derivatives of X_i are given by

$$(2.12) \quad X_{i|\beta} = X_{i|j} B_\beta^j + X_i|_j N^j H_\beta, \quad X_i|_\beta = X_i|_j B_\beta^j.$$

A. Rapcsák [11] introduced three kinds of hyperplanes. M. Matsumoto [8] obtained their characteristic conditions, which are given in the following lemmas:

Lemma 2.1. *A hypersurface F^{n-1} is a hyperplane of first kind if and only if $H_\alpha = 0$ or equivalently $H_0 = 0$.*

Lemma 2.2. *A hypersurface F^{n-1} is a hyperplane of second kind if and only if $H_{\alpha\beta} = 0$.*

Lemma 2.3. *A hypersurface F^{n-1} is a hyperplane of third kind if and only if $H_{\alpha\beta} = 0 = M_{\alpha\beta}$.*

3 The Finsler space ${}^*F^n = (M^n, {}^*L)$

Let us denote $b_i y^i$ by β , then indicatory property of h_{ij} yield $\dot{\partial}_i \beta = b_i$. The quantities corresponding to ${}^*F^n$ is denoted by asterisk over that quantity. We shall use following notations $L_i = \dot{\partial}_i L = l_i$, $L_{ij} = \dot{\partial}_i \dot{\partial}_j L$, $L_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L$. From (1.3), we get

$$(3.1) \quad {}^*L_{ij} = e^\tau (1 + \rho - \tau) L_{ij} + \frac{e^\tau}{L} m_i m_j,$$

$$(3.2) \quad \begin{aligned} {}^*L_{ijk} = & e^\tau (1 + \rho - \tau) L_{ijk} + (\rho - \tau) \frac{e^\tau}{L} [m_i L_{jk} + m_j L_{ik} + m_k L_{ij}] \\ & - \frac{e^\tau}{L^2} [m_j m_k l_i + m_i m_k l_j + m_i m_j l_k - m_i m_j m_k], \end{aligned}$$

where $\tau = \frac{\beta}{L}$, $m_i = b_i - \tau l_i$. The normalised supporting element and the metric tensor of ${}^*F^n$ are obtained as [2]

$$(3.3) \quad {}^*l_i = e^\tau (m_i + l_i),$$

$$(3.4) \quad {}^*g_{ij} = \nu e^{2\tau} g_{ij} + e^{2\tau} (2\tau^2 - \tau - \rho) l_i l_j + e^{2\tau} (1 - 2\tau) (b_i l_j + b_j l_i) + 2e^{2\tau} b_i b_j.$$

Differentiating the angular metric tensor h_{ij} with respect to y^k , we get

$$\dot{\partial}_k h_{ij} = 2C_{ijk} - \frac{1}{L} (l_i h_{jk} + l_j h_{ik}),$$

which gives

$$(3.5) \quad L_{ijk} = \frac{2}{L} C_{ijk} - \frac{1}{L^2} (h_{ij} l_k + h_{jk} l_i + h_{ki} l_j).$$

Using this, the equation (3.2) may be re-written as

$$(3.6) \quad {}^*C_{ijk} = \nu e^{2\tau} C_{ijk} + \frac{2}{L} e^{2\tau} m_i m_j m_k + \frac{1}{2L} e^{2\tau} (2\nu - 1) (m_i h_{kj} + m_j h_{ki} + m_k h_{ij}),$$

where $\nu = 1 + \rho - \tau$.

The inverse metric tensor of ${}^*F^n$ is derived as follows[2]:

$$(3.7) \quad {}^*g^{ij} = \frac{e^{-2\tau}}{\nu} \left[g^{ij} - \frac{1}{m^2 + \nu} b^i b^j + \frac{\tau - \nu}{m^2 + \nu} (b^i l^j + b^j l^i) - \left\{ \frac{\tau - \nu}{m^2 + \nu} (m^2 + \tau) - \rho \right\} l_i l_j \right],$$

where b is magnitude of the vector $b^i = g^{ij} b_j$.

The relation between cartan connection coefficients of ${}^*F^n$ and F^n is given by

$$(3.8) \quad {}^*F_{jk}^i = F_{jk}^i + D_{jk}^i.$$

The expressions for D_{00}^i , D_{0k}^i and D_{jk}^i are given by [2]

$$(3.9) \quad \begin{aligned} D_{00}^i = & \frac{L}{\nu e^\tau} \left[\frac{e^\tau}{L} \beta_{|0} m^i + 2e^\tau F_0^i \right] + l^i \left[E_{00} - \frac{L}{e^\tau} (m^2 + \nu)^{-1} \left(\frac{e^\tau}{L} \beta_{|0} m^2 + 2e^\tau F_{\beta 0} \right) \right] \\ & - \frac{m^i L}{\nu e^\tau} (m^2 + \nu)^{-1} \left[\frac{e^\tau}{L} \beta_{|0} m^2 + 2e^\tau F_{\beta 0} \right], \end{aligned}$$

$$(3.10) \quad D_{0j}^i = \frac{L G_j^i}{\nu e^\tau} + \frac{l^i}{e^\tau} \left[G_j - L (m^2 + \nu)^{-1} G_{\beta j} \right] - \frac{m^i L}{\nu e^\tau} (m^2 + \nu)^{-1} G_{\beta j},$$

$$(3.11) \quad D_{ik}^j = \frac{LH_{ik}^j}{\nu e^\tau} + \frac{l^j}{e^\tau} \left[H_{ik} - L(m^2 + \nu)^{-1} H_{\beta ik} \right] - \frac{m^j L}{\nu e^\tau} (m^2 + \nu)^{-1} H_{\beta ik},$$

where

$$(3.12) \quad \begin{aligned} 2G_{ij} = & \frac{e^\tau}{L} (\beta_{|j} m_i - \beta_{|i} m_j) + 2e^\tau F_{ij} - \nu e^\tau L_{ijr} D_{00}^r - A_{ijk} y^k - \frac{e^\tau}{L^2} m_r m_i m_j D_{00}^r \\ & + \frac{(\nu - 1)}{L} e^\tau \beta_{|0} L_{ij} + \frac{e^\tau}{L^2} B_0 m_i m_j + e^\tau \rho_0 L_{ij}, \end{aligned}$$

$$(3.13) \quad G_j = e^\tau (E_{j0} - F_{j0}),$$

$$(3.14) \quad \begin{aligned} 2H_{jik} = & -\nu e^\tau \left[L_{ijr} D_{0k}^r + L_{jkr} D_{0i}^r - L_{kir} D_{0j}^r \right] + A_{jki} + A_{kij} - A_{ijk} \\ & - \frac{e^\tau}{L^2} \left[m_i m_j m_r D_{0k}^r + m_j m_k m_r D_{0i}^r - m_k m_i m_r D_{0j}^r \right] \\ & + (\nu - 1) \frac{e^\tau}{L} (\beta_{|k} L_{ij} + \beta_{|i} L_{jk} - \beta_{|j} L_{ki}) + e^\tau \left[\rho_k L_{ij} + \rho_i L_{jk} - \rho_j L_{ki} \right] \\ & + \frac{e^\tau}{L^2} \left[\beta_{|k} m_i m_j + \beta_{|i} m_j m_k - \beta_{|j} m_k m_i \right], \end{aligned}$$

$$(3.15) \quad \begin{aligned} 2H_{ik} = & \frac{e^\tau}{L} (\beta_{|k} m_i + \beta_{|i} m_k) + e^\tau E_{ik} - \left[\nu e^\tau L_{ir} + \frac{e^\tau}{L} m_i m_r \right] D_{0k}^r \\ & - \left[\nu e^\tau L_{kr} + \frac{e^\tau}{L} m_k m_r \right] D_{0i}^r, \end{aligned}$$

$$(3.16) \quad A_{ijk} = \frac{e^\tau}{L} D_{0k}^r \mathfrak{S}_{(rij)} \left[(\nu - 1) m_r L_{ij} - \frac{m_i m_j l_r}{L} \right].$$

Here we have used $m_i m^i = m^2 = m^i b_i$ and $H_{ik}^j = g^{jm} H_{mik}$. Also we note that $E_{00} = E_{ij} y^i y^j = b_{i|j} y^i y^j = (b_i y^i)_{|j} y^j = \beta_{|0}$, $F_0^i = g^{ij} F_{j0}$. The subscript ‘0’ denotes the contraction by supporting element y^i , unless otherwise stated. The subscript ‘ β ’ denotes the contraction by supporting element b^i . $\mathfrak{S}_{(ijk)}$ denote cyclic interchange of indices i, j, k and summation.

Now, we state a Lemma which are used later.

Lemma 3.1. [6] *If the h -vector b_i is gradient then the scalar ρ is constant.*

4 The Hypersurface ${}^*F^{n-1}$ of the space ${}^*F^n$

Let us consider Finslerian hypersurfaces $F^{n-1} = (M^{n-1}, \underline{L}(u, v))$ of F^n and ${}^*F^{n-1} = (M^{n-1}, {}^*\underline{L}(u, v))$ of ${}^*F^n$. Let N^i be the unit normal vector at a point of F^{n-1} . The

functions $B_\alpha^i(u)$ may be considered as component of $(n-1)$ linearly independent vectors tangent to F^{n-1} and they are invariant under h-exponential change of Finsler metric. The unit normal vector ${}^*N^i(u, v)$ of ${}^*F^{n-1}$ is uniquely determined by

$$(4.1) \quad {}^*g_{ij}B_\alpha^i{}^*N^j = 0, \quad {}^*g_{ij}{}^*N^i{}^*N^j = 1.$$

The inverse projection factors ${}^*B_i^\alpha(u, v)$ of B_α^i along ${}^*F^{n-1}$ are defined as

$$(4.2) \quad {}^*B_i^\alpha = {}^*g^{\alpha\beta}{}^*g_{ij}B_\alpha^j,$$

where ${}^*g^{\alpha\beta}$ is the inverse of metric tensor ${}^*g_{\alpha\beta}$ of ${}^*F^{n-1}$.

From (4.2), it follows that

$$(4.3) \quad B_\alpha^i{}^*B_i^\beta = \delta_\alpha^\beta, \quad B_\alpha^i{}^*N_i = 0, \quad {}^*N_iB_\alpha^i = 0, \quad {}^*N_i{}^*N^i = 1,$$

and further

$$(4.4) \quad B_\alpha^i{}^*B_j^\beta + {}^*N^i{}^*N_j = \delta_j^i.$$

Now, Transvection of (2.3) by v^α gives

$$(4.5) \quad y_j N^j = 0.$$

Transvecting (3.4) by $N^i N^j$ and by using (4.5), we have

$$(4.6) \quad {}^*g_{ij}N^i N^j = \nu e^{2\tau} + 2e^{2\tau}(b_i N^i)^2,$$

this implies that

$$(4.7) \quad \frac{N^j}{e^\tau \sqrt{\nu + 2(b_i N^i)^2}}$$

is unit vector.

Also, Transvecting (3.4) by $B_\alpha^i N^j$ and using (4.5), gives us

$$(4.8) \quad {}^*g_{ij}B_\alpha^i N^j = (b_j N^j) e^{2\tau} \left\{ (1-2\tau) l_i B_\alpha^i + 2b_i B_\alpha^i \right\}.$$

This shows that N^j is normal if and only if R.H.S. of equation (4.8) is zero. Since $e^{2\tau} \left\{ (1-2\tau) l_i B_\alpha^i + 2b_i B_\alpha^i \right\}$ can not be zero, otherwise transvection of $e^{2\tau} \left\{ (1-2\tau) l_i B_\alpha^i + 2b_i B_\alpha^i \right\}$ by v^α gives $L = 0$, which is not possible. Hence N^j is normal to ${}^*F^{n-1}$ if and

only if $b_j N^j = 0$.

From (4.7) and (4.8), we may state that

$$(4.9) \quad {}^*N^i = \frac{N^i}{e^\tau \sqrt{\nu}}$$

is unit normal vector of ${}^*F^{n-1}$.

Which in view of (3.4) and (4.5), gives

$$(4.10) \quad {}^*N_i = N_i e^\tau \sqrt{\nu}.$$

Thus, we have :

Theorem 4.1. *Let ${}^*F^n$ be the Finsler space obtained from F^n by h-exponential change given by (1.3). Further if ${}^*F^{n-1}$ and F^{n-1} are the hypersurfaces of these spaces. Then the vector b_i is tangential to hypersurface F^{n-1} if and only if every vector normal to F^{n-1} is also normal to ${}^*F^{n-1}$. And then the normal vector is given by (4.9).*

Let b_i is gradient vector, i.e. $b_{j|i} = b_{i|j}$, then

$$(4.11) \quad F_{ij} = 0,$$

which in view of Lemma (3.1), gives

$$(4.12) \quad \rho_i = 0.$$

Now, if b_i is tangent to hyperplane F^{n-1} i. e.

$$(4.13) \quad b_j N^j = 0.$$

Using (4.5), (4.11) and (4.13), we have

$$(4.14) \quad D_{00}^i N_i = 0.$$

The normal curvature tensor ${}^*H_\alpha$ for hypersurface ${}^*F^{n-1}$ is given by

$${}^*H_\alpha = {}^*N_i (B_{0\alpha}^i + {}^*G_j^i B_\alpha^j),$$

by use of (2.8) and (4.9), above equation becomes

$$(4.15) \quad {}^*H_\alpha = \sqrt{\nu} e^\tau \left(H_\alpha + N_i D_{0j}^i B_\alpha^j \right),$$

which on transvection by v^α and using (4.14), gives

$$(4.16) \quad {}^*H_0 = \sqrt{\nu} e^\tau H_0.$$

Thus in view of Lemma (2.1), we have :

Theorem 4.2. *Let the h -vector b_i be a gradient and tangent to hypersurface F^{n-1} . Then the hypersurface F^{n-1} is a hyperplane of first kind if and only if hypersurface ${}^*F^{n-1}$ is hyperplane of first kind.*

Taking the relative h -covariant differentiation of (4.13) with respect to the Cartan connection of F^{n-1} , we get

$$b_{i|j}N^j + b_iN^i_{|\beta} = 0.$$

Using (2.11) and (2.12), the above equation gives

$$(b_{i|j}B^j_{|\beta} + b_i|_jN^jH_{|\beta})N^i - b_iH_{\alpha\beta}B^{\alpha}_jg^{ij} = 0.$$

Travecting by v^{β} and using (2.9), we get

$$b_{i|0}N^i = (H_{\alpha} + M_{\alpha}H_0)B^{\alpha}_jb^j - b_i|_jH_0N^iN^j.$$

For the hypersurface to be first kind, $H_0 = 0 = H_{\alpha}$. Then above equation reduces to $b_{i|0}N^i = 0$. If the vector b_i is gradient, *i.e.* $b_{i|j} = b_j|i$, then we get

$$E_{i0}N^i = b_{i|0}N^i = \beta_iN^i.$$

The tensors D^i_{00} , D^i_{0j} , G_{ij} and G_j satisfies the following, which can be easily verified :

$$(4.17) \quad \begin{aligned} D^i_{00}N_i &= 0, \quad D^r_{0j}L_{jr}N^j = 0 \\ L_{ijr}D^r_{00} &= (E_{00} - (m^2 + \nu)^{-1}\beta_0m^2) \left[\left(\frac{2\rho}{L^2\nu} - \frac{1}{L^2} \right) h_{ij} - \frac{1}{L^2\nu} (m_jl_i + m_il_j) \right], \\ G_{ij}N^iB^j_{\alpha} &= 0, \quad D^i_{0j}N_iB^j_{\alpha} = 0, \quad G_jN^j = 0, \quad G_{ij}b^iN^j = 0, \\ D^i_{0j}b_iN^j &= 0, \quad D^i_{0j}N^jB^k_{\alpha}h_{ik} = 0, \quad D^r_{0j}l_rN^j = 0, \quad G^r_jl_rN^j = 0. \end{aligned}$$

The second fundamental h - tensor ${}^*H_{\alpha\beta}$ for hyperplane ${}^*F^{n-1}$ is given by

$${}^*H_{\alpha\beta} - {}^*M_{\alpha}{}^*H_{\beta} = {}^*N_i(B^i_{\alpha\beta} + {}^*F^i_{jk}B^j_{\alpha}B^k_{\beta}),$$

then by use of (2.7), (3.8) and (4.10), above equation gives

$$(4.18) \quad {}^*H_{\alpha\beta} - {}^*M_{\alpha}{}^*H_{\beta} = e^{\tau}\sqrt{\nu} \left[H_{\alpha\beta} + N_iD^i_{jk}B^j_{\alpha}B^k_{\beta} \right] - e^{\tau}\sqrt{\nu}M_{\alpha}H_{\beta}.$$

Contracting (3.11) by $B_\alpha^i B_\beta^k N_j$ and using $m^j N_j = 0$, $l^j N_j = 0$, we get

$$D_{ik}^j B_\alpha^i B_\beta^k N_j = \frac{L}{\nu e^\tau} H_{ik}^j B_\alpha^i B_\beta^k N_j = -\frac{L}{2\nu e^\tau} H_{jik} N^j B_\alpha^i B_\beta^k,$$

which in view of (3.14) and (4.17), gives

$$(4.19) \quad D_{ik}^j B_\alpha^i B_\beta^k N_j = -\frac{L}{2} \left[L_{ijr} D_{0k}^r + L_{jkr} D_{0i}^r - L_{kir} D_{0j}^r \right] N^j B_\alpha^i B_\beta^k.$$

Now we calculate each terms of the above equation separately.

Transvecting (3.12) by N^j , we have

$$(4.20) \quad G_{ij} N^j = \mu N_i,$$

where

$$\mu = \frac{1}{2L^2} \left[-e^\tau (E_{00} - (m^2 + \nu)^{-1} \beta_0 m^2) (2\rho - \nu) - e^\tau \nu D_{00}^r m_r + \nu(\nu - 1) e^\tau \beta_0 \right].$$

Contracting L_{ijr} by $N^j B_\alpha^i B_\beta^k D_{0k}^r$ and using (1.2), (3.10) and above equation, we obtain

$$(4.21) \quad L_{kir} N^j B_\alpha^i B_\beta^k D_{0j}^r = \frac{2\mu}{\nu e^\tau} M_{\alpha\beta},$$

Transvecting L_{ijr} by $N^j B_\alpha^i B_\beta^k$ and using (1.2), (3.10) and (3.12), we get

$$(4.22) \quad L_{ijr} N^j B_\alpha^i B_\beta^k D_{0k}^r = \frac{2}{\nu e^\tau} \left[\lambda M_{\alpha\beta} - \frac{e^\tau}{2L} \beta_r C_{ij}^r N^j B_\alpha^i B_\beta^k m_k \right],$$

where

$$\lambda = \frac{1}{2L^2} \left[-e^\tau (E_{00} - (m^2 + \nu)^{-1} \beta_0 m^2) (2\rho - \nu) - e^\tau \nu (\nu - 1) D_{00}^s m_s + \nu(\nu - 1) e^\tau \beta_0 \right].$$

Similarly, transvecting L_{kjr} by $N^j B_\alpha^i B_\beta^k D_{0i}^r$ and using $M_{\alpha\beta} = M_{\beta\alpha}$, we have

$$(4.23) \quad L_{kjr} N^j B_\alpha^i B_\beta^k D_{0i}^r = \frac{2}{\nu e^\tau} \left[\lambda M_{\alpha\beta} - \frac{e^\tau}{2L} \beta_r C_{ij}^r N^j B_\beta^i B_\alpha^k m_k \right].$$

Plugging (4.21), (4.22), (4.23) in equation (4.19), we obtain

$$(4.24) \quad D_{ik}^j N_j B_\alpha^i B_\beta^k = \frac{L(\mu - 2\lambda)}{e^\tau \nu} M_{\alpha\beta} + \frac{e^\tau}{2L} \beta_r C_{ij}^r \left[N^j B_\alpha^i B_\beta^k m_k + N^j B_\beta^i B_\alpha^k m_k \right].$$

Now, suppose that h -vector b_i satisfies the condition

$$(4.25) \quad b_{r|0} C_{ij}^r = \kappa h_{ij},$$

then

$$(4.26) \quad \beta_r C_{ij}^r = \kappa h_{ij} ,$$

where κ is a scalar function.

So, using $h_{ij} B_\alpha^i N^j = 0$, equation (4.24) yields

$$(4.27) \quad D_{ik}^j B_\alpha^i B_\beta^k N_j = \frac{L(\mu - 2\lambda) M_{\alpha\beta}}{\nu e^\tau} .$$

And then (4.18) becomes

$$(4.28) \quad {}^*H_{\alpha\beta} - {}^*M_\alpha {}^*H_\beta = e^\tau \sqrt{\nu} \left[H_{\alpha\beta} + \frac{L(\mu - 2\lambda)}{\nu e^\tau} M_{\alpha\beta} \right] - e^\tau \sqrt{\nu} M_\alpha H_\beta .$$

Next, transvecting (3.6) by $B_\alpha^i B_\beta^j N^k$ and using (2.10), we have

$$(4.29) \quad {}^*M_{\alpha\beta} = \sqrt{\nu} e^\tau M_{\alpha\beta} .$$

Thus from (2.28) and (2.29), we have:

Theorem 4.3. *For the exponential change with an h -vector, let the h -vector b_i be a gradient and tangential to hypersurface F^{n-1} and satisfies condition (4.25). Then*

1. *${}^*F^{n-1}$ is a hyperplane of second kind if F^{n-1} is hyperplane of second kind and $M_{\alpha\beta} = 0$.*
2. *${}^*F^{n-1}$ is a hyperplane of third kind if F^{n-1} is hyperplane of third kind.*

5 Example

A Finsler space F^n is called *P -Finsler space if the $(v)hv$ -torsion tensor P_{ij}^r satisfies

$$(5.1) \quad P_{ij}^r := C_{ij|0}^r = \lambda C_{ij}^r .$$

Taking h -covariant derivative of (1.1) and using $L_{|k} = 0 = h_{ij|k}$ and $\rho_i = 0$, we get

$$(5.2) \quad b_{r|k} C_{ij}^r + b_r C_{ij|k}^r = 0 .$$

Contracting the above equation by y^k and using (5.1), we get

$$b_{r|0} C_{ij}^r + \lambda b_r C_{ij}^r = 0 ,$$

which in view of (1.1), becomes

$$(5.3) \quad b_{r|0} C_{ij}^r = \kappa h_{ij} , \quad \kappa = -\frac{\lambda \rho}{L} ,$$

which is required condition (4.25). Thus, we have:

Theorem 5.1. *For the exponential change with an h -vector, let the h -vector b_i be a gradient and tangential to hypersurface F^{n-1} of a $*P$ -Finsler space F^n . Then*

1. *$*F^{n-1}$ is a hyperplane of second kind if F^{n-1} is hyperplane of second kind and $M_{\alpha\beta} = 0$.*
2. *$*F^{n-1}$ is a hyperplane of third kind if F^{n-1} is hyperplane of third kind.*

A Landsberg space is $*P$ -Finsler space for $\kappa = 0$.

Thus, we have :

Corollary 5.1. *For the exponential change with an h -vector, let the h -vector b_i be a gradient and tangential to hypersurface F^{n-1} of a Landsberg space F^n . Then*

1. *$*F^{n-1}$ is a hyperplane of second kind if F^{n-1} is hyperplane of second kind and $M_{\alpha\beta} = 0$.*
2. *$*F^{n-1}$ is a hyperplane of third kind if F^{n-1} is hyperplane of third kind.*

Discussion

Gupta and Pandey [3] have proved that for Kropina change with an h -vector (let the h -vector b_i be a gradient and tangential to hypersurface F^{n-1} and satisfies condition $\beta_r C_{ij}^r = 0$),

$*F^{n-1}$ is a hyperplane of third kind if F^{n-1} is hyperplane of third kind.

In present paper, authors proved that for exponential change with an h -vector (same conditions),

$*F^{n-1}$ is a hyperplane of third kind if F^{n-1} is hyperplane of third kind.

Notice that Kropina change with an h -vector is finite in nature (in the sense that number of terms) whereas exponential change with an h -vector is infinite in nature, although in both cases (finite and infinite) same result holds.

The question is that *Is there any particular type of change with an h -vector (same conditions) for which $*F^{n-1}$ is a hyperplane of third kind if F^{n-1} is hyperplane of third kind ?*

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